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Model-Following System with Assignable Error Dynamics and Its Application to Aircraft

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A main objective of this paper is to propose a practical tool for designing a model-following system. Successive differentiations of output errors between a linear time-invariant system (a plant) and a model yield controllable output error dynamics in a block-decoupled companion form. Such a particular form of the error dynamics makes it easy to determine a model-following control law such that the output error dynamics have arbitrarily assigned stability characteristics. Assigning asymptotic stability to the output error dynamics alone does not guarantee system stability. The transmission zeros which are inherent to the input-output relations must be in the left-hand half-plane for internal system stability. The approach has been applied to a variable stability and response aircraft under development. An example of the longitudinal model-following system on this aircraft is shown to demonstrate the proposed method.

Nomenclature

A = system matrix
A^* = matrix defined by Eq. (21)
B = control matrix
B^* = coefficient matrix of the highest order in the numerator polynomials of transfer function matrix [Eq. (21)]
C = output matrix
c_k = the kth output vector
$e_i(t)$ = the ith error phase variable
$\underline{e}_k(t)$ = extended error vector for the kth output [Eq. (11)]
$e(t)$ = extended output error vector [Eq. (22)]
$\bar{f}_k(t)$ = forcing function to the kth output error dynamics [Eq. (14)]
$G(s)$ = transfer function matrix of a compensator
H_τ = parameter for thrust response, 1/s
H_{δ_T} = throttle effectiveness, kg/s·mm
i = index
J_k = companion matrix associated with the kth output error dynamics [Eq. (18)]
K = matrix to determine the output error dynamics = diagonal $\{k_i^T\}$
K_x = feedback gain matrix [Eq. (27)]
K_{xm} = feedforward gain matrix [Eq. (27)]
K_{um} = direct link gain matrix [Eq. (27)]
k_k^T = row vector to assign the kth output error dynamics [Eq. (15)]
M_k = matrix to define the kth output error vector [Eq. (10)]
M = matrix to define the output error dynamics = $\{M_k\}$
m = dimension of control vector
n = dimension of state vector
r = dimension of output vector
s = Laplace's parameter
t = time
$U(s)$ = Laplace transformation of control input vector
$u(t)$ = control vector
$x(t)$ = state vector
$y(t)$ = output vector
$y_k(t)$ = the kth output
α_i = coefficient of characteristic polynomial
$\Delta_i(s;k_i^T)$ = characteristic equation for the ith output error dynamics [Eq. (19)]
$\Delta_e(s;K)$ = characteristic equation of the output error dynamics [Eq. (29)]
δ_T = throttle position, mm
σ_i = relative order of the ith output
τ = thrust, kg

Subscripts and Superscripts

e = error
i = index
k = index
m = model
T = transpose of vectors and matrices

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Introduction

THE pole assignment problem of linear time-invariant systems has long been of primary interest to system designers and has been studied extensively. Zeros of transfer functions have often received secondary concern except when necessary for the closed-loop system stability. However, the zeros are also quite important in characterizing the system behavior. The problem of assigning a plant (a linear time-invariant system) the desired zeros as well as the poles is the "model-following problem" or the "model-matching problem" where an attempt is made to find a certain control law such that the plant outputs follow the desired model outputs for *any* control inputs.

Among other approaches for designing model-following systems, a linear-quadratic optimization approach[1-7] seems to be the most popular and traditional. Although this approach is quite practical, the selection of weighting matrices is a basic problem in the application.

There is an approach[8,9] using a set of feedback invariants and a coordinate transformation for solving an exact-model-matching problem. This approach considers model-matching only by state feedback.

An algorithmic approach[10] has been developed which directly considers the transfer functions in terms of Markov moments. This approach is based upon the structure algorithm that the authors[10] developed and covers a broad class of problems.

Recently, a geometrical approach[11] has been applied to model-following problems. This approach utilizes highly sophisticated mathematics and provides precise solutions to a broad class of model-following problems. This seems to be an analysis-oriented approach.

Existing approaches for designing model-following control systems have their own merits. However, it would be appreciated by practical engineers if there were an approach that neither contained somewhat obscure and roundabout design parameters nor required complicated transformations and algorithms.

This paper proposes a new, mathematically simple, but practically useful, design method for a model-following system. The basic concept is to control not only the output errors between a given plant and a model but also their derivatives, so that the inevitable time delays of output responses to control inputs may be compensated. In contrast with other approaches, a strong point of the proposed method is that the output error dynamics are directly exposed to designers and are arbitrarily assignable. Furthermore, it completely determines a major part of the closed-loop dynamics. Hence, it is extremely easy for practical designers to intervene in the design process and to blend their engineering insights and experiences with mathematical requirements.

A Beechcraft Model 65 (Queen Air) has been remodeled into a variable stability and response aircraft (VSRA) to investigate operations and flying qualities of STOL aircraft in the terminal area. The proposed design method of the model-following system was applied to the basic control system design of the VSRA. This paper shows an example of the longitudinal model-following system of the VSRA to demonstrate the proposed method. A model to be followed in this example is similar to that of a Boeing 747. Through this practical application, the proposed method has been evaluated to be quite simple and practically useful for designing a model-following system.

Model-Following Control Law

This section describes the model-following problem and derives a model-following control law in a straightforward way. Although this paper deals with strictly proper systems, the same approach can be extended to proper systems. (When the order of the numerator polynomial of a transfer function is strictly less than or equal to that of the denominator, the system is called "strictly proper" or "proper," respectively.)

A completely controllable and observable linear time-invariant system, which is called a plant, is given by

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad y(t) = Cx(t) \tag{1}$$

where $x(t)$ is an n-dimensional state vector, $u(t)$ an m-dimensional control vector, and $y(t)$ an r-dimensional output vector; A, B, and C are constant matrices with appropriate dimensions. $(\dot{\ })$ indicates the time derivative throughout this paper.

Any strictly proper desired model to be followed is represented by

$$\dot{x}_m(t) = A_m x_m(t) + B_m u_m(t)$$
$$y_m(t) = C_m x_m(t) \tag{2}$$

where $x_m(t)$ is an n_m-dimensional state vector, $u_m(t)$ an m_m-dimensional control vector, and $y_m(t)$ an r_m-dimensional output vector; A_m, B_m, and C_m are constant matrices with appropriate dimensions.

Sometimes, desired input-output relations may be given by a transfer function matrix. If this is the case, Eq. (2) should be a minimum realization of the desired model.

The problem is to find a control law $u(t)$ that makes the plant output $y(t)$ follow the desired output $y_m(t)$ for *any* input $u_m(t)$. Obviously, the dimension of $y(t)$ and $y_m(t)$ must be the same, i.e., $r = r_m$. By Laplace transformation, let the transfer functions of Eqs. (1) and (2) be

$$\frac{Y}{U}(s) = C(sI_n - A)^{-1}B \qquad \frac{Y_m}{U_m}(s) = C_m(sI_{n_m} - A_m)^{-1}B_m$$

where s is the Laplace's parameter, I_n and I_{n_m} are an $n \times n$ and an $n_m \times n_m$ identity matrices, respectively.
Then, the problem seems to ask for $U(s) = \mathcal{L}[u(t)]$ such that

$$\frac{Y}{U}(s) \cdot U(s) = \frac{Y_m}{U_m}(s) \cdot U_m(s) \qquad \text{for any} \quad U_m(s) \tag{3}$$

By substitution of $U(s) = G(s) \cdot U_m(s)$

$$\frac{Y}{U}(s) \cdot G(s) = \frac{Y_m}{U_m}(s) \tag{4}$$

The problem is equivalent to finding a precompensator $G(s)$. However, it is not as simple as that since there are a number of ways to produce the same effect as $G(s)$. It is understood that one wants to find a control law with a good combination of feedforward and feedback controls.

There are usually some time delays between the outputs and the inputs, and it is important to apply control inputs as early as possible to achieve the desired outputs. A key concept presented in this paper is to control not only the output errors but also their derivatives in order to compensate for the inevitable time delays of output responses to control inputs. This concept plays an important role in the subsequent development with respect to how many integrations of inputs are necessary for outputs to start responding for the first time. Thus, the relative order of each output transfer function is defined at the start. By writing the kth row of matrix C as c_k, consider a sequence of row vectors c_kB, $c_kAB, \ldots,$ $c_kA^{i-1}B, \ldots,$ and let σ_k be the minimum i with which $c_kA^{i-1}B$ becomes nonzero for the first time. Namely,

$$\sigma_k = \min\{i; c_kA^{i-1}B \neq 0\} \qquad (k = 1, 2, \ldots, r) \tag{5a}$$

Likewise, for the model given by Eq. (2)

$$\sigma_{mk} = \min\{j; c_{mk}A_m^{j-1}B_m \neq 0\} \qquad (k = 1, 2, \ldots, r) \tag{5b}$$

where c_{mk} is the kth row of matrix C_m.
The σ_k defined as above is called the relative order of the kth output since, under the assumption of Eq. (5a), the kth row of the transfer function matrix is written as follows[12-14]

$$\frac{Y_k}{U}(s) = c_k(sI_n - A)^{-1}B = \frac{c_kA^{\sigma_k-1}Bs^{n-\sigma_k} + c_k(A^{\sigma_k} + \alpha_1A^{\sigma_k-1})Bs^{n-\sigma_k-1} + \ldots - \alpha_n c_kA^{-1}B}{s^n + \alpha_1s^{n-1} + \alpha_2s^{n-2} + \ldots + \alpha_{n-1}s + \alpha_n} \tag{6}$$

Namely, the kth output starts to respond for the first time to inputs after σ_k times of integrations. It is assumed for simplicity throughout this paper that

$$\sigma_k \leq \sigma_{mk} \qquad (k = 1, 2, \ldots, r) \tag{7}$$

Furthermore, d_k is defined as follows just for the sake of convenience:

$$d_k = \sum_{i=1}^{k} \sigma_i \qquad (k=1,2,\ldots,r) \tag{8}$$

As has been mentioned, it is important to express the errors and their derivatives between the plant output $y(t)$ and the model output $y_m(t)$ in a set of controllable dynamic equations with respect to the plant control $u(t)$. Then, a control law is found so that the output error dynamics have arbitrarily assigned stability characteristics.

Let $e_1(t)$ be the error between $y_1(t)$ and $y_{m1}(t)$

$$e_1(t) \triangleq y_1(t) - y_{m1}(t) = c_1 x(t) - c_{m1} x_m(t)$$

Differentiate $e_1(t)$ once and use Eqs. (1) and (2). Since $c_1 B$ and $c_{m1} B_m$ are both zeros if $\sigma_1 \neq 1$, one obtains

$$\dot{e}_1(t) = c_1 A x(t) - c_{m1} A_m x_m(t) \triangleq e_2(t)$$

Defining a new error phase variable $e_2(t)$ as above, differentiate $e_2(t)$ and use Eqs. (1-3):

$$\dot{e}_2(t) = c_1 A^2 x(t) - c_{m1} A_m^2 x_m(t) \triangleq e_3(t)$$

If the differentiation of $e_2(t)$ does not yield $u(t)$ explicitly on the right-hand side, another new phase variable $e_3(t)$ is introduced. Continuing this procedure until the plant control $u(t)$ explicitly appears on the right-hand side, one finally obtains the following expression after σ_1 times differentiations of $e_1(t)$.

$$\dot{e}_{\sigma_1}(t) = c_1 A^{\sigma_1} x(t) - c_{m1} A_m^{\sigma_1} x_m(t) + c_1 A^{\sigma_1} B u(t) - c_{m1} A_m^{\sigma_1 - 1} B_m u_m(t)$$

Knowing $e_{\sigma_1}(t) \equiv e_{d_1}(t)$ from Eq. (8), the error between $y_2(t)$ and $y_{m2}(t)$ is defined as

$$e_{d_1+1}(t) \triangleq y_2(t) - y_{m2}(t) = c_2 x(t) - c_{m2} x_m(t)$$

The successive differentiation of $e_{d_1+1}(t)$ and the definitions of new error phase variables are carried out in the same way as for $e_1(t)$, and will terminate at σ_2 times differentiations of $e_{d_1+1}(t)$. Generally, the error of the kth output is defined as

$$e_{d_1+1}(t) \triangleq y_2(t) - y_{m2}(t) = c_2 x(t) - c_{m2} x_m(t)$$

By Eq. (5a), the successive differentiation of the above yields

$$\dot{e}_{d_{k-1}+1}(t) = c_k A x(t) - c_{mk} A_m x_m(t) \triangleq e_{d_{k-1}+2}(t)$$

$$\dot{e}_{d_{k-1}+2}(t) = c_k A^2 x(t) - c_{mk} A_m^2 x_m(t) \triangleq e_{d_{k-1}+3}(t)$$

$$\vdots \qquad\qquad \vdots$$

$$\dot{e}_{d_k - 1}(t) = c_k A^{\sigma_k - 1} x(t) - c_{mk} A_m^{\sigma_k - 1} x_m(t) \triangleq e_{d_k}(t)$$

$$\dot{e}_{d_k}(t) = c_k A^{\sigma_k} x(t) - c_{mk} A_m^{\sigma_k} x_m(t) + c_k A^{\sigma_k - 1} B u(t) - c_{mk} A_m^{\sigma_k - 1} B_m u_m(t) \tag{9}$$

For the simplicity of notation, define M_k, M_{mk}, and $\underline{e}_k(t)$ as follows

$$M_k \triangleq \begin{bmatrix} c_k \\ c_k A \\ \vdots \\ c_k A^{\sigma_k - 1} \end{bmatrix} \qquad M_{mk} \triangleq \begin{bmatrix} c_{mk} \\ c_{mk} A_m \\ \vdots \\ c_{mk} A_m^{\sigma_k - 1} \end{bmatrix} \tag{10}$$

$$\underline{e}_k(t) \triangleq [e_{d_{k-1}+1}(t),\ e_{d_{k-1}+2}(t),\ \ldots,\ e_{d_k - 1}(t),\ e_{d_k}(t)]^T \tag{11}$$

where $[\cdot]^T$ indicates the transpose of vector or matrix throughout the paper. Then, the error vector of the kth output is written as follows

$$\underline{e}_k(t) = M_k x(t) - M_{mk} x_m(t) \tag{12}$$

The output error dynamics of the kth element [Eq. (9)] can be expressed in a vector form as

$$\dot{\underline{e}}_k(t) = \left[\begin{array}{ccccc} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ \hline 0 & \cdots & \cdots & \cdots & 0 \end{array}\right] \underline{e}_k(t) + \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ \hline c_k A^{\sigma_k - 1} B \end{array}\right] u(t) + \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ \hline f_k(t) \end{array}\right] \tag{13}$$

where

$$f_k(t) \triangleq c_k A^{\sigma_k} x(t) - c_{mk} A_m^{\sigma_k} x_m(t) - c_{mk} A_m^{\sigma_k - 1} B_m u_m(t) \tag{14}$$

Necessary and sufficient conditions must be found that makes $\underline{e}_k(t)$ asymptotically converge for any initial conditions and any $u_m(t)$. Defining an σ_k dimensional row vector k_k^T as

$$k_k^T \triangleq [k_{k1},\ k_{k2},\ \ldots,\ k_{k\sigma_k}] \tag{15}$$

add and subtract $k_k^T \underline{e}_k(t)$ in the last row of Eq. (13). Then, the right-hand side may be formally split into a homogeneous part with a companion form and a forcing part. Assume that the plant control $u(t)$ is chosen to always eliminate the forcing part, i.e.,

$$c_k A^{\sigma_k - 1} B u(t) + k_k^T \underline{e}_k(t) + f_k(t) = 0 \tag{16}$$

Then, the homogeneous part becomes

$$\dot{\underline{e}}_k(t) = J_k \underline{e}_k(t) \qquad \underline{e}_k(0) = \text{arbitrary} \tag{17}$$

where J_k is an $\sigma_k \times \sigma_k$ matrix of companion form defined by

$$J_k \triangleq \left[\begin{array}{ccccc} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ \hline & & -k_k^T & & \end{array}\right] \tag{18}$$

Since the characteristic equation of Eq. (17) is obtained as

$$\Delta_k(s;k_k^T) \triangleq s^{\sigma_k} + k_{k\sigma_k} s^{\sigma_k - 1} + k_{k\sigma_k - 1} s^{\sigma_k - 2} + \ldots + k_{k2} s + k_{k1} = 0 \tag{19}$$

an appropriate selection of row vector k_k^T can provide any desired asymptotic stability to the kth output error dynamics. From Eq. (16), the plant control $u(t)$ that always eliminates all the forcing terms for all $k=1,2,\ldots,r$ must satisfy

$$B^* u(t) + K\underline{e}(t) + A^* x(t) - A_m^* x_m(t) - B_m^* u_m(t) = 0 \tag{20}$$

where

$$K=\left[\begin{array}{c|c|c|c} k_1^T & 0 & & 0 \\ 0 & k_2^T & & \vdots \\ \vdots & 0 & \cdots & \vdots \\ \vdots & \vdots & & 0 \\ 0 & 0 & & k_r^T \end{array}\right] \quad (r\times d_r)$$

$$B^* \triangleq \begin{bmatrix} c_1A^{\sigma_1-1}B \\ c_2A^{\sigma_2-1}B \\ \vdots \\ c_rA^{\sigma_r-1}B \end{bmatrix} (r\times r) \qquad A^* \triangleq \begin{bmatrix} c_1A^{\sigma_1} \\ c_2A^{\sigma_2} \\ \vdots \\ c_rA^{\sigma_r} \end{bmatrix} (r\times n) \tag{21}$$

and matrices $A_m^*\,(r\times n_m)$ and $B_m^*\,(r\times m_m)$ are also defined in the same way. $\underline{e}(t)$ is the output error vector of dimension $d_r\,(=\sigma_1+\sigma_2+\ldots+\sigma_r)$ that is defined by

$$\underline{e}(t) \triangleq [e_1^T(t) \,|\, \underline{e}_2^T(t) \,|\, \ldots \,|\, \underline{e}_r^T(t)]^T \tag{22}$$

The use of Eq. (12) expresses $\underline{e}(t)$ in terms of the plant and the model states as follows

$$\underline{e}(t) \triangleq \begin{bmatrix} M_1 \\ \hline M_2 \\ \hline \vdots \\ \hline M_r \end{bmatrix} x(t) - \begin{bmatrix} M_{m1} \\ \hline M_{m2} \\ \hline \vdots \\ \hline M_{mr} \end{bmatrix} x_m(t) \equiv Mx(t) - M_m x_m(t) \tag{23}$$

Then, the substitution of Eq. (23) into Eq. (20) yields

$$B^*u(t) + (A^*+KM)x(t) - (A_m^*+KM_m)x_m(t) - B_m^*u_m(t) = 0 \tag{24}$$

It can be shown that the above equation is equivalent to Eq. (3).[15] Hence, the conditions for the existence of $u(t)$ satisfying Eq. (24) are expected to be the same as those for the existence of a right-integral inverse of the given plant if the model dynamics to be followed are completely arbitrary, except $\sigma_k \le \sigma_{mk}$ for all k. One of the necessary conditions for the existence of $u(t)$ is obviously that the plant must have at least as many inputs as outputs to be controlled in arbitrary ways, i.e., $m\ge r$. For simplicity, however, it is assumed here that $r=m$ unless otherwise stated. Furthermore, the matrix $B^*\,(r\times m)$ is assumed to have the maximum rank $r\,(=m)$:

$$\text{rank}[B^*]=r \qquad \det\{B^*\} \equiv |B^*| \neq 0 \tag{25}$$

Therefore, Eq. (24) can be solved with respect to $u(t)$ as follows

$$u(t) = -K_x x(t) + u_c(t) \qquad u_c(t) = K_{xm}x_m(t) + K_{um}u_m(t) \tag{26}$$

where

$$K_x \triangleq B^{*-1}(A^*+KM) \qquad K_{xm} \triangleq B^{*-1}(A_m^*+KM_m)$$
$$K_{um} \triangleq B^{*-1}B_m^* \tag{27}$$

The block diagram of the above control law is given in Fig. 1. The control $u(t)$ of Eq. (26) is a typical real-model-following control law and consists of the feedback control $-K_x x(t)$ and the feedforward control $u_c(t)$ which contains the desired model dynamics. Under such a control law as Eq. (26), the output error dynamics becomes homogeneous and a *block-decoupled companion* form results:

$$\begin{bmatrix} \dot{\underline{e}}_1(t) \\ \hline \dot{\underline{e}}_2(t) \\ \hline \vdots \\ \hline \dot{\underline{e}}_r(t) \end{bmatrix} = \begin{bmatrix} J_1 & & & \mathbf{0} \\ & J_2 & & \\ & & \ddots & \\ \mathbf{0} & & & J_r \end{bmatrix} \begin{bmatrix} \underline{e}_1(t) \\ \hline \underline{e}_2(t) \\ \hline \vdots \\ \hline \underline{e}_r(t) \end{bmatrix} \tag{28}$$

The characteristic equation of the complete error dynamics then becomes the polynomial with the order of $d_r\,(=\sigma_1+\sigma_2+\ldots+\sigma_r)$ in s as follows

$$\Delta_e(s;K) \triangleq \prod_{i=1}^{r} \Delta_i(s;k_i^T) \tag{29}$$

where $\Delta_i(s;k_i^T)$ are polynomials defined by Eq. (19).

Thus, the characteristic roots of the output error dynamics can be arbitrarily assigned by an appropriate selection of matrix K. Those should be assigned so that the output error dynamics of Eq. (28) are asymptotically stable. Then, as long as the control law satisfies Eqs. (26) and (27), the plant outputs will asymptotically follow the model outputs for any initial conditions (errors) no matter what model inputs $u_m(t)$ are applied.

The asymptotic stability of the output error dynamics does not guarantee the internal system stability, so that the model-following system sometimes may not be realizable even if the output errors formally seem to diminish. Accordingly, conditions under which the system with the model-following control law of Eq. (26) is internally stable and practically realizable must be investigated. Only the stability of the feedback control part (the closed-loop) of the system is of interest since, from Eq. (26), the feedforward control part depends only upon the characteristics of the desired model to be followed. By substituting Eq. (26) into Eq. (1), the closed-loop dynamics are given by

$$\dot{x}(t) = (A-BK_x)x(t) + Bu_c(t) \qquad y(t) = Cx(t) \tag{30}$$

Then, the characteristic equation can be obtained as follows

$$\Delta_c(s;K) \triangleq |sI_n - A + BK_x|$$
$$= |S(s;K)|\,|sI_n - A|\,|C(sI_n-A)^{-1}B|\,/\,|B^*| \tag{31}$$

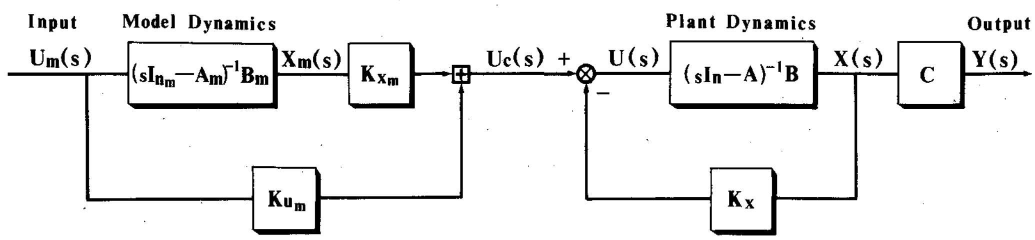

Fig. 1 Block diagram of model-following system.

where $S(s;K)$ is an $r\times r$ diagonal matrix in polynomial field given by

$$S(s;K) \triangleq \mathrm{diag}\{\Delta_i(s;k_i^T)\} \tag{32}$$

The determinant of the transfer function matrix of a given plant [Eq. (1)] can be written as[14]

$$\left|\frac{Y}{U}(s)\right| \equiv |C(sI_n-A)^{-1}B| = \frac{\phi(s)}{|sI_n-A|} \tag{33}$$

where $\phi(s)$ is a polynomial in s, which is also obtained by[16]

$$\phi(s) = \left|\begin{array}{c|c} sI_n-A & B \\ \hline -C & 0 \end{array}\right| \tag{34}$$

The roots of $\phi(s)=0$ are often called transmission zeros[13-16] and play important roles, although they are usually hidden behind the conventional zeros of every element of the transfer function matrix $Y/U(s)$. Then, by Eqs. (29, 31-33), the characteristic equation becomes

$$\Delta_c(s;K) = \Delta_e(s;K)\phi(s)/|B^*| = 0 \tag{35}$$

Thus, d_r poles out of n closed-loop poles are exactly the same as those of the output error dynamics which can be arbitrarily assigned through the matrix K. The rest of the poles are the same as the $(n-d_r)$ transmission zeros, defined by $\phi(s)=0$, which are known to be feedback-invariant and inherent for a given plant. If the matrix B^* is nonzero but singular, the number of transmission zeros is not $(n-d_r)$. An extended B^* must be newly defined by further differentiations of output errors. The resulting control law may require some derivatives of $u_m(t)$.

Therefore, all the transmission zeros must have negative real parts so that the feedback loop is internally stable and practically realizable. This may be a relatively strong condition on the plant for model-following. On the model, however, no particular conditions have been imposed except the condition of Eq. (7), i.e., $Y_m/U_m(s)$ is arbitrary. As is expected from Eq. (4), it is shown that the model-following system with the control law of Eqs. (26) and (27) implicitly contains an integral inverse system of the given plant.[15] By pole-zero cancellation, the closed-loop system is shown to be not completely observable any longer if there are transmission zeros. Thus, the transmission zeros are as significant for such a model-following system as the assigned stability of the output error dynamics.

Finally, it is noted that, although the matrix K does not need to take a block-diagonal form as in Eq. (21), this paper has assumed it to be of that form for simplicity.

The output error dynamics are usually not available to designers in most of model-following control system designs. It must, therefore, be hard to conjecture how the output errors are going to behave. In this approach, however, the output error dynamics are apparent to the designer in the explicit form of Eq. (28), and they are completely governed by a matrix K which is the only design matrix left to the designer. Since the output error dynamics are of a block-decoupled companion form, a unique matrix K can easily be determined by the characteristic root assignment. It has been shown that assigning the characteristic roots of the output error dynamics is nothing other than assigning a subset of the closed-loop poles of the plant. Thus, in contrast with other approaches, a great advantage of this approach is the fact that designers can easily intervene in the design process and blend their engineering judgments and experiences with mathematical requirements through the selection of a matrix K.

An Example: Application to Aircraft Longitudinal Motion

The proposed design method of model-following system described in the preceding section was applied to the basic control system design of the 747 VSRA described earlier. This section shows an example on the preliminary system design[17] about the longitudinal motion of the VSRA. The linearized aircraft longitudinal equations of motion[18] are assumed to be

$$\begin{bmatrix}\dot u\\ \dot w\\ \dot q\\ \dot\theta\\ \dot\tau\end{bmatrix} = \begin{bmatrix} X_u & X_w & 0 & -g\cos\gamma_0 & X_\tau\\ Z_u & Z_w & U_0 & -g\sin\gamma_0 & 0\\ M'_u & M'_w & M'_q & -gM_{\dot w}\sin\gamma_0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & H_\tau\end{bmatrix}\begin{bmatrix}u\\ w\\ q\\ \theta\\ \tau\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & 0\\ M'_{\delta e} & 0\\ 0 & 0\\ 0 & H_{\delta_T}\end{bmatrix}\begin{bmatrix}\delta_e\\ \delta_T\end{bmatrix} \tag{36a}$$

The stability axis system is used and, with a few exceptions, the notations are conventional. In particular, τ indicates the engine thrust assumed to have a first-order delay, and δ_T is the throttle lever position. For the derivatives of pitching moment, $M'_{(\cdot)} \triangleq M_{(\cdot)} + M_{\dot w}Z_{(\cdot)}$ except $M'_q \triangleq M_q + M_{\dot w}U_0$. The dynamics of a desired model (aircraft) to be followed are assumed for simplicity to be given by the same equation as Eq. (36a). The corresponding quantities are indicated with subscript m as $(X_u)_m$, γ_m, δ_{em}, etc. Since the original aircraft has two controls, only two plant outputs can be controlled to follow the desired model outputs. Various combinations of two outputs are conceivable as the outputs for the model-following. In this example, the velocity u and the flight path angle γ responses are considered as the outputs.

$$y(t) = \begin{bmatrix}u\\ \gamma\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0 & 0\\ 0 & -1/U_0 & 0 & 1 & 0\end{bmatrix}[u\ w\ q\ \theta\ \tau]^T \tag{36b}$$

Although u and γ are important quantities to determine the flight trajectory, the attitude θ and the angle of attack $\alpha = w/U_0$ should not be too far from those of the desired model. These will be checked by simulation runs.

The method is applied as follows:

1) Determine the location of the transmission zeros and check if the model-following system is realizable. From Eqs. (34) and (36),

$$\phi(s) = X_\tau H_{\delta_T} M'_{\delta_e}(Z_w - (g/U_0)\sin\gamma_0) \neq 0$$

Therefore, there are no transmission zeros and the model-following control is practical. However, if Z_{δ_e} is not assumed to be zero, two transmission zeros result. One is usually located to the far right of the origin in the Laplace plane and the other to the left for conventional aircraft. For the VSRA, the effects of Z_{δ_e} are considered to be negligibly small.

2) The relative orders σ_i are calculated and then B^* is evaluated. For the velocity u, Eq. (5a) yields $c_1B = 0$ and

$$c_1AB = [X_u,\ X_w,\ 0,\ -g\cos\gamma_0,\ X_\tau]B$$
$$= [0,\ X_\tau H_{\delta_T}] \neq 0$$

Then, $\sigma_1=2$. Likewise, for the flight path angle γ, $c_2B=c_2AB=0$ and

$$c_2A^2B=[-M_{\delta_e}(Z_w-(g/U_0)\sin\gamma_0), -(Z_u/U_0)X_\tau H_{\delta_T}]\neq 0$$

So, $\sigma_2=3$. Thus, $d_2=\sigma_1+\sigma_2=5=n$. This coincides with the fact that there are no transmission zeros as examined above. The matrix B^* is obtained as

$$B^*=\begin{bmatrix} 0 & X_\tau H_{\delta_T} \\ -M'_{\delta_e}(Z_w-(g/U_0)\sin\gamma_0) & -(Z_u/U_0)X_\tau H_{\delta_T}\end{bmatrix}$$

and is obviously nonsingular:

$$|B^*|=X_\tau H_{\delta_T}M'_{\delta_e}(Z_w-(g/U_0)\sin\gamma_0)=\phi(s)$$

Hence, the model-following technique derived in this paper can directly be applied to the longitudinal system design of the VSRA.

3) Since $d_1=\sigma_1=2$, the output errors are defined as

$$e_1(t)\triangleq u-u_m \qquad e_3(t)\triangleq\gamma-\gamma_m$$

Since $\sigma_1=2$ and $\sigma_2=3$, the five phase variables of the output errors are defined by $\underline{e}(t)=Mx(t)-M_mx_m(t)$ as Eqs. (10) and (23).

$$\begin{bmatrix} e_1(t)\\ e_2(t)\\ \hdashline e_3(t)\\ e_4(t)\\ e_5(t)\end{bmatrix}\equiv\begin{bmatrix} u-u_m\\ \dot u-\dot u_m\\ \hdashline \gamma-\gamma_m\\ \dot\gamma-\dot\gamma_m\\ \ddot\gamma-\ddot\gamma_m\end{bmatrix}\equiv\begin{bmatrix} c_1\\ c_1A\\ \hdashline c_2\\ c_2A\\ c_2A^2\end{bmatrix}x(t)-\begin{bmatrix} c_{m1}\\ c_{m1}A_m\\ \hdashline c_{m2}\\ c_{m2}A_m\\ c_{m2}A_m^2\end{bmatrix}x_m(t)$$

The phase variables have been defined up to the longitudinal acceleration error and the flight path angular acceleration error. The matrices M and M_m can be easily calculated analytically and numerically. Then, after the addition and the subtraction of $K\underline{e}(t)$, the output error dynamics become

$$\begin{bmatrix} \dot e_1(t)\\ \dot e_2(t)\\ \hdashline \dot e_3(t)\\ \dot e_4(t)\\ \dot e_5(t)\end{bmatrix}=\begin{bmatrix} 0 & 1 & & \mathbf{0} & \\ -k_{11} & -k_{12} & & & \\ \hdashline & & 0 & 1 & 0\\ \mathbf{0} & & 0 & 0 & 1\\ & & -k_{21} & -k_{22} & -k_{23}\end{bmatrix}\begin{bmatrix} e_1(t)\\ e_2(t)\\ \hdashline e_3(t)\\ e_4(t)\\ e_5(t)\end{bmatrix}+\begin{bmatrix} 0\\ c_1AB\\ \hdashline 0\\ 0\\ c_2A^2B\end{bmatrix}\begin{bmatrix}\delta_e\\ \delta_T\end{bmatrix}+\begin{bmatrix} 0\\ f_1(t)\\ \hdashline 0\\ 0\\ f_2(t)\end{bmatrix}+\begin{bmatrix} 0 & 0\\ 1 & 0\\ \hdashline 0 & 0\\ 0 & 0\\ 0 & 1\end{bmatrix}K\underline{e}(t)$$

where $f_i(t)$ are given by Eq. (14) and

$$K=\begin{bmatrix} k_{11} & k_{12} & 0 & 0 & 0\\ \hdashline 0 & 0 & k_{21} & k_{22} & k_{23}\end{bmatrix}$$

4) The control law $u(t)$ is determined so as to eliminate the forcing terms in the above equations. Substituting the contexts of $f_1(t)$ and $f_2(t)$, the equation that corresponds to Eq. (24) becomes

$$\underbrace{\begin{bmatrix} c_1AB\\ c_2A^2B\end{bmatrix}}_{B^*}u(t)+\underbrace{\begin{bmatrix} c_1A^2\\ c_2A^3_m\end{bmatrix}}_{A^*}x(t)-\underbrace{\begin{bmatrix} c_{m1}A^2_m\\ c_{m2}A^3_m\end{bmatrix}}_{A^*_m}x_m(t)-\underbrace{\begin{bmatrix} c_{m1}A_mB_m\\ c_{m2}A^2_mB_m\end{bmatrix}}_{B^*_m}u_m(t)+K\underline{e}(t)=0$$

By the substitution of $\underline{e}(t)\triangleq Mx(t)-M_mx_m(t)$, the control law $u(t)$ is obtained as Eqs. (26) and (27):

$$u(t)=-B^{*-1}(A^*+KM)x(t)+u_c(t)$$

$$u_c(t)=B^{*-1}(A^*_m+KM_m)x_m(t)+B^{*-1}B^*_mu_m(t)$$

The feedback and the feedforward gains can be computed once the matrix K is appropriately chosen. But K is still left open to the designer.

5) Under the model-following control law, the output error dynamics become homogeneous and the characteristic equation $\Delta_e(s;K)$ is given by

$$\Delta_e(s;K)=\Delta_u(s;k_1^T)\Delta_\gamma(s;k_2^T)$$

where $\Delta_u(s;k_1^T)$ and $\Delta_\gamma(s;k_2^T)$ are given by

$$\Delta_u(s;k_1^T)=s^2+k_{12}s+k_{11} \qquad \sigma_1=2$$

$$\Delta_\gamma(s;k_2^T)=s^3+k_{23}s^2+k_{22}s+k_{21} \qquad \sigma_2=3$$

and correspond to the velocity and the flight path angle error dynamics, respectively. Since there are no transmission zeros and $\phi(s)=|B^*|$ in this example, Eq. (35) yields

$$\Delta_c(s;K)\equiv\Delta_e(s;K)$$

Namely, the matrix K completely determines not only the output error dynamics but also the closed-loop stability. Hence, the closed-loop system may be regarded as an aircraft with a stability augmentation system (SAS).

The model to be followed by the original aircraft (Queen Air) is assumed to be a similar aircraft to a Boeing 747 in the landing configuration. Stability derivatives[19] used in this example are shown in Table 1. Using these values, a design matrix K has to be determined by specifying the characteristic roots of the output error dynamics or the closed-loop dynamics. Although the root assignment is completely arbitrary, large deviations from the original characteristic roots should be avoided in order for the feedback and the feedforward gains to be moderate. Therefore, it has been assumed that the closed-loop characteristics of the VSRA still maintain the feature of the original aircraft, i.e., the phugoid and the short-period modes of motion are well separated from each other. By engineering judgment, the characteristic equations have been specified as

$$\Delta_u(s;k_1^T)=s^2+\sqrt{2}(0.3191)s+(0.3191)^2$$

$$\Delta_\gamma(s;k_2^T)=(s+1)[s^2+\sqrt{2}(3.191)s+(3.191)^2]$$

The design matrix K can be easily computed to satisfy the above equations. The corresponding gain matrices are given in Table 2.

Figure 2 shows the time histories of air speed $u(t)$, angle of attack $\alpha(t)$, pitch attitude $\theta(t)$, and flight path angle $\gamma(t)$, respectively, to an elevator step input of -2 deg (pull). The desired responses to be followed (responses similar to the Boeing 747) are shown in broken lines. The full lines indicate the responses of the VSRA designed by the proposed technique presented in this paper.

When there are no initial misalignments between the plant and model states as in this figure, the model-following outputs u and γ of the VSRA perfectly follow those of the model, so that the broken lines of u and γ are invisible. The pitch attitude and the angle of attack, which are of secondary interest in this example, do not follow those of the model, but they respond within practically acceptable errors. Such errors in secondary quantities may be a drawback of this approach for model-following. Since the VSRA has only two independent controls (elevator and throttle), only two outputs of the plant (u and γ) have been matched with the model's at the expense of some errors in θ and α. If the VSRA were equipped with the third independent control such as a direct lift control device, the quantities of secondary interest (θ and α) could also be matched with the model responses.

Figure 2 also indicates the effects of some plant parameter variations. Plant parameters have been varied by keeping the feedback and the feedforward gains constant. The effects of Z_{δ_e} were quite small and invisible in the figure, so the assumption of $Z_{\delta_e}=0$ in this example was justified. The responses of the VSRA with a half of the nominal M'_w (i.e., $0.5\ M'_w$) are shown in chain lines. The effects of M'_w variation turned out to be very small for the velocity and the angle of attack, while the pitch attitude and the flight path angle responses were subjected to about half a degree of variation from those of the nominal VSRA. The broken long lines show the effects of the X_u variation ($0.5\ X_u$). The flight path angle response was not affected so much, and cannot be distinguished from that of the nominal VSRA. On the other hand, the velocity response suffered considerable deviation from the nominal VSRA's. However, it should be noticed that 50% variations of M'_w and X_u are fairly large in comparison with normal accuracy in stability derivatives. Hence, it is considered that there would be no practical difficulties with these errors. Furthermore, these errors are due to parameter uncertainties and would be decreased considerably by adopting a so-called type I system which could simply be

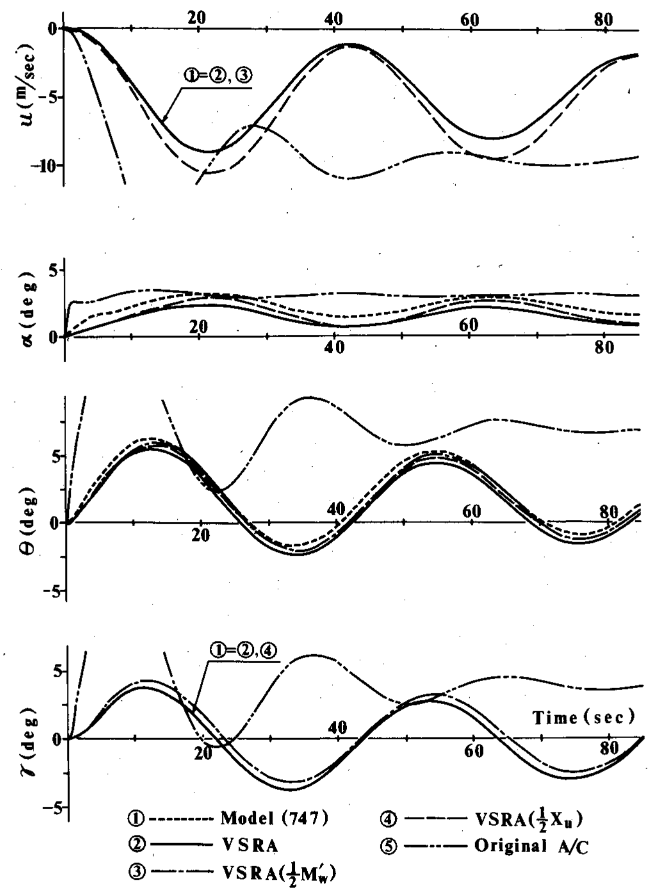

Fig. 2 Longitudinal responses and parameter sensitivities of VSRA to step elevator input $\delta_{me}=-2$ deg.

Table 1 Stability derivatives

	Beech 65	Boeing 747
U_0, m/s	53.6	67.36
γ_0, deg	-6.0	0.0
X_u, 1/s	-0.0922	-0.0433
X_w, 1/s	0.102	0.05196
X_τ, m/kg·s	0.00294	3.76×10^{-5}
Z_u, 1/s	-0.366	-0.272
Z_w, 1/s	-1.281	-0.4896
M'_u, 1/s·m	0.00245	-0.000257
M'_w, 1/s·m	-0.0871	-0.0054
M'_q, 1/s	-1.659	-0.412
$M_{\dot w}$, 1/m	-0.0067	-0.000816
H_τ, 1/s	-1.0	-0.50
$M'_{\delta e}$, 1/s	-7.83	-0.376
$H_{\delta T}$, kg/s·mm	40.0	1000.0

Table 2 Feedback and feedforward gains

	u, m/s	w, m/s	q, rad/s	θ, rad	τ, kg
K_x –					
$K_{\delta e(\cdot)}$[a]	-0.00641	0.00765	-0.329	-0.986	-7.68×10^{-6}
$K_{\delta T(\cdot)}$	0.268	-0.797	-36.6	-28.9	-0.0160
	u_m, m/s	w_m, m/s	q_m, rad/s	θ_m, rad	τ_m, kg
K_{xm} –					
$K_{\delta ec(\cdot)}$	-0.00484	0.00646	-0.230	-1.01	-7.12×10^{-8}
$K_{\delta TC(\cdot)}$	0.595	-0.0361	-53.6	-34.0	-2.95×10^{-5}
	δ_{em}, rad	δ_{TM}, mm			
K_{um} –					
$K_{\delta ec(\cdot)}$	0.0187	1.06×10^{-5}			
$K_{\delta TC(\cdot)}$	0.0	0.320			

[a] $K_{\delta e(\cdot)}$ indicates the gain from $(\cdot)$ to δ_e. The others are similarly defined.

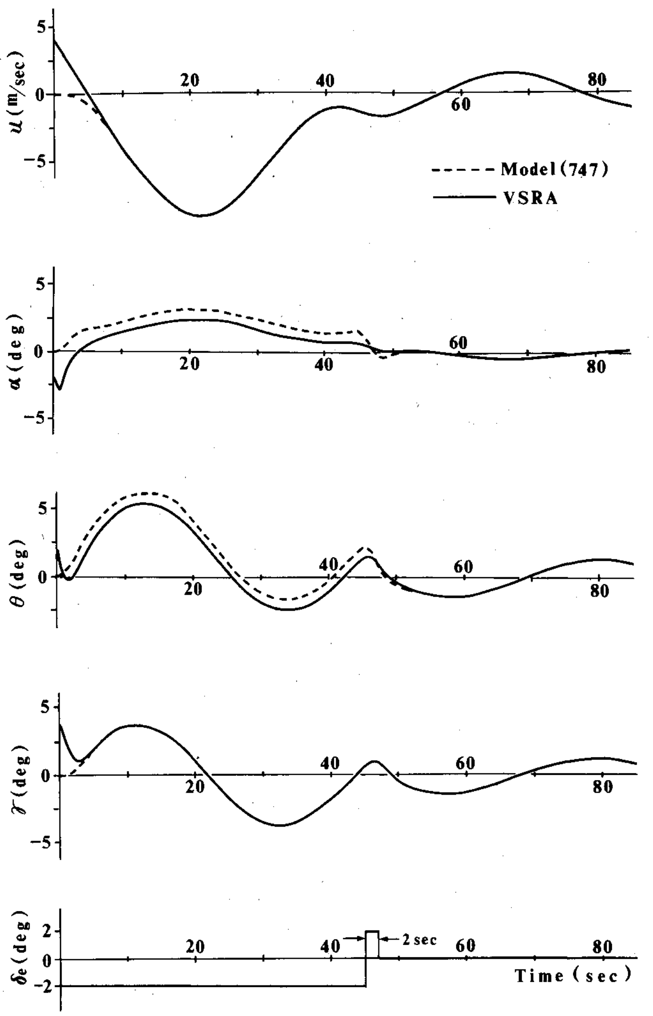

Fig. 3 Effects of initial misalignment and control variation on longitudinal responses of VSRA.

designed by the method of state augmentation. The error dynamics may also be adjusted through the matrix K so as to minimize the sensitivities to parameter variations.[6] The responses of the original aircraft (Queen Air) to the same input are also indicated in Fig. 2 for reference.

In Fig. 3, the model is initially set in a trimmed state and the VSRA in an off-trimmed state: $u(0)=4$ m/s, $\alpha(0)=-2$ deg, $\theta(0)=2$ deg, and $\gamma(0)=4$ deg. A step elevator input of -2 deg (pull) were applied for the first 45 s, and then a down-elevator control of 2 deg for the next 2 s. As is expected, the model-following outputs u and γ of the VSRA follow those of the model very well after small transient errors which are characterized by the specified output error dynamics. It is also seen that, after the transient errors have almost decayed, the plant outputs always follow the model outputs whatever control is applied thereafter. These substantiate the homogeneous output error dynamics in the theory developed earlier.

Simulation runs have been carried out for cases of other parameter uncertainties, various sensor errors, type I system, etc.[17] In the long run, the proposed approach was concluded to provide a practically useful system design tool.

Conclusions

This paper has proposed a practically applicable new design method of the model-following system in which the output error dynamics are arbitrarily assignable. A basic concept of the method is to control not only the output errors but also their derivatives in order to compensate the inevitable time delays of output responses to control inputs.

The output error dynamics were characterized by a particular "block-decoupled companion form," the order of which was shown to depend upon the sum of the relative orders. A real model-following control law was determined so as to provide the output error dynamics a desirable asymptotic stability. The closed-loop characteristic roots were shown to consist of the roots of the assigned output error dynamics and the transmission zeros. It is one of the great advantages of this method that the assignment of the output error dynamics is nothing but the assignment of a subset of the closed-loop poles. However, the transmission zeros must be in the left-hand half-plane for internal system stability. This may be a relatively strong constraint on the plant for the realization of model-following system. However, this requirement yields the desirable feature that model dynamics may be arbitrarily assigned under the constraints among the relative orders.

In the numerator polynomials of a plant transfer function matrix, the coefficient matrix B^* of the highest order was assumed to be nonsingular. It will be shown, however, that the proposed method is still applicable to systems with a singular B^* by considering derivatives of model inputs or certain precompensators.

All the states between the plant and the model cannot be matched with the proposed method except for particular cases, since the number of independent controls is usually less than that of the states. The plant outputs must carefully be chosen so that secondary quantities are within their tolerance limits during the system's operation. Furthermore, how to assign the output error dynamics is a problem still to be solved, and must completely depend upon the designers' engineering judgments.

The proposed method was applied to the system design for the variable stability and response aircraft (VSRA). An example on the longitudinal model-following system of the VSRA was shown to demonstrate the proposed method. Extensive studies on the basic system design including simulation runs appraised the proposed method for model-following to work out quite well in practice.

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